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THE METHOD OF LIE SERIES IN THE MOTION-SEPARATION PROBLEM IN NONLINEAR MECHANICS*

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A method is proposed for separating the motions for multifrequency systems in standard form, based on the construction of the Hamiltonian form of the system by introducing adjoint variables, and subsequent use of the group of canonical transformations generated by a Lie generator. The method yields a simpler algorithm for constructing higher approximations compared with the well-known Krylov-Bogolyubov method /1,2/, since the transformation of $n+m$ equations is replaced by a transformation of one scalar function, while instead of the $n+m$ equations of the change of variables a scalar equation is set up for the Lie generator and its partuclar solution is derived in closed form. It is shown that the introduction of adjoint variables does not lead to an increase in the dimensions of the problem. A definition of resonance in the i-th approximation is introduced and a resonance form of the method is given. An example is presented.

Consider the equations of motion of multifrequency systems in the following standard form

$$\varphi^{\cdot} = \omega(I) + \varepsilon F(\varphi, I), \quad I^{\cdot} = \varepsilon G(\varphi, I), \quad \varphi = (\varphi_1, \ldots, \varphi_n) \tag{1}$$
$$I = (I_1, \ldots, I_m)$$

where ε is a small parameter, and the right-hand sides are (2π)-periodic in all φ_i and are analytic in some domain. The motion-separation problem /1,2/ consists in finding a replacement $(\varphi, I) \to (\psi, J)$ whose implementation reduces Eqs.(1) to the form

$$\psi^{\cdot} = \omega(J) + \varepsilon P(J, \varepsilon), \quad J^{\cdot} = \varepsilon Q(J, \varepsilon) \tag{2}$$

In this system the equations for the slow variables are separated and can be investigated independently. To reduce system (1) to form (2), we construct a Hamiltonian form of the problem /3/ by introducing the variables u, v adjoint to φ, I such that the Hamiltonian can be written as

$$H(\varphi, I, u, v, \varepsilon) = \omega(I) u + \varepsilon [F(\varphi, I) u + G(\varphi, I) v] \tag{3}$$

We shall seek the change of variables solving the motion-separation problem in the class of canonical replacements, which reduces the Hamiltonian (3) to a form corresponding to Eq.(2). Unlike Poincare's method /4/ we shall construct the canonical replacement not by means of a generating function but using a Lie generator /3,5/. The advantage of such an approach is that it enables us to obtain the replacement equations in explicit form right away, without having to solve them, as is unavoidable when using the generating function.

The Lie generator is the Hamiltonian function $S(\varphi, I, u, v, \varepsilon)$ of some auxiliary Hamiltonian system

$$\frac{d\varphi}{d\tau} = \frac{\partial S}{\partial u}, \quad \frac{dI}{d\tau} = \frac{\partial S}{\partial v}, \quad \frac{du}{d\tau} = -\frac{\partial S}{\partial \varphi}, \quad \frac{dv}{d\tau} = -\frac{\partial S}{\partial I}$$

(τ is a certain new independent variable not having the meaning of time). Suppose that the general solution of this system is known:

$$\varphi, I, u, v \quad \text{are functions of} \quad \psi, J, p, q, \tau, \varepsilon \tag{4}$$

where (ψ, J, p, q) are the initial values of (φ, I, u, v) for $\tau = 0$. Functions (4) can be looked upon as a one-parameter Lie group of canonical replacements $(\varphi, I, u, v) \to (\psi, J, p, q)$ of the phase variables. These replacements will be used to transform the Hamiltonian (3). The infinitesimal operator of group (4) has the form

$$U = \frac{\partial S}{\partial p}\frac{\partial}{\partial \psi} + \frac{\partial S}{\partial q}\frac{\partial}{\partial J} - \frac{\partial S}{\partial \psi}\frac{\partial}{\partial p} - \frac{\partial S}{\partial J}\frac{\partial}{\partial q} \tag{5}$$
$$S(\psi, J, p, q) = S(\varphi, I, u, v)|_{\tau=0}$$

According to the well-known Lie theorem the function (4) defining the group can be written as the following series (the Lie series):

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$$\varphi = \psi + \tau U\psi + \frac{\tau^2}{2!} U^2\psi + \ldots \equiv e^{\tau U}\psi \equiv \psi + \tau\{\psi, S\} + \frac{\tau^2}{2!}\{\{\psi, S\}, S\} + \ldots \quad (6)$$

$$I = J + \tau UJ + \frac{\tau^2}{2!} U^2 J + \ldots \equiv e^{\tau U} J \equiv J + \tau\{J, S\} + \frac{\tau^2}{2!}\{\{J, S\}, S\} + \ldots$$

where {. . .} are Poisson brackets. The canonical replacements (4) (or (6)) lead to the following transformation of the Hamiltonian:

$$K(\psi, J, p, q, \varepsilon) \equiv H[\varphi(\psi, J, p, q, \varepsilon, \tau), I(\ldots), u(\ldots), v(\ldots), \varepsilon] = \quad (7)$$
$$H + \tau UH + \frac{\tau^2}{2!} U^2 H + \ldots, \quad H = H(\psi, J, p, q, \varepsilon)$$

From among all the transformations of group (4) (or (6)) we choose a single one from the condition $\tau = \varepsilon$; then instead of (7) we obtain

$$K(\psi, J, p, q, \varepsilon) = H(\psi, J, p, q, \varepsilon) + \varepsilon\{H, S\} + \frac{\varepsilon^2}{2!}\{\{H, S\}, S\} + \ldots \quad (8)$$

The generator S, as well as the transformed Hamiltonian K, will be sought in the form of power series in ε

$$S(\psi, J, p, q, \varepsilon) = S_1 + S_2\varepsilon + \ldots, \quad K(\psi, J, p, q, \varepsilon) = K_0 + K_1\varepsilon + \ldots \quad (9)$$

Substituting (3) and (9) into (8) and separating the orders, and also hearing in mind that $H_0 = \omega(J)p$, $H_1 = F(\psi, J)p + G(\psi, J)q$, we obtain

$$K_0 = \omega(J)p, \quad K_1 = p\frac{d\omega}{dJ}\frac{\partial S_1}{\partial q} - \omega\frac{\partial S_1}{\partial \psi} + F(\psi, J)p + G(\psi, J)q \quad (10)$$
$$K_2 = p\frac{d\omega}{dJ}\frac{\partial S_2}{\partial q} - \omega\frac{\partial S_2}{\partial \psi} + \{H_1, S_1\} + \frac{1}{2}\{\{H_0, S_1\}, S_1\}$$
$$K_3 = p\frac{d\omega}{dJ}\frac{\partial S_3}{\partial q} - \omega\frac{\partial S_3}{\partial \psi} + \{H_1, S_2\} + \frac{1}{2}\{\{H_0, S_2\} +$$
$$\{H_1, S_1\}, S_1\} + \frac{1}{2}\{\{H_0, S_1\}, S_2\} + {}^1/_6\{\{\{H_0, S_1\}, S_1\}, S_1\} + , \ldots$$

Relations (10) specify the connection between the Hamiltonian of the original problem (1) and the Hamiltonian of the problem transformed using replacement (4), where S is an as yet unknown generator.

We shall seek the Lie generator S from the condition for excluding the variable ψ: $H(\varphi, I, u, v, \varepsilon) \to K(J, p, q, \varepsilon)$ from the transformed Hamiltonian. For this we write K_1

$$K_1(J, p, q) = \lim_{\theta\to\infty}\frac{1}{\theta}\int_0^\theta [F(\omega\theta, J)p + G(\omega\theta, J)q]\,d\theta \quad (11)$$

and we represent the perturbed part of the Hamiltonian in the form

$$F(\psi, J)p + G(\psi, J)q = K_1(J, p, q) + f^1(\psi, J)p + g^1(\psi, J)q \quad (12)$$

Substituting (12) into (10), we obtain, in coordinate-wise form

$$\sum_{i=1}^n\sum_{j=1}^m p_i\frac{\partial\omega_i}{\partial J_j}\frac{\partial S_1}{\partial q_j} + \sum_{i=1}^n\left(-\omega_i\frac{\partial S_1}{\partial\psi_i} + f_i^1 p_i\right) + \sum_{j=1}^m g_j^1 q_j = 0 \quad (13)$$

This partial differential equation enables us to determine S_1. A particular solution of this equation yields the following quadrature:

$$S_1 = \frac{1}{\omega_n}\sum_{i=1}^n\int\left[f_i^1\left(\frac{\omega_1}{\omega_n}\psi_n + r_1, \ldots, \frac{\omega_{n-1}}{\omega_n}\psi_n + \right.\right. \quad (14)$$
$$\left.\left. r_{n-1}, \psi_n, J\right)p_i\,d\psi_n + \frac{1}{\omega_n}\sum_{j=1}^m\int g_j^1(\ldots)q_j\,d\psi_n + \right.$$
$$\frac{1}{\omega_n^2}\sum_{i=1}^n\sum_{j=1}^m p_i\frac{\partial\omega_i}{\partial J_j}\iint g_j^1(\ldots)\,d\psi_n^2$$

After performing the integration in (14), r_k $(k = 1, \ldots, n-1)$ should be replaced by $r_k = \psi_k - (\omega_k/\omega_n)\psi_n$. Henceforth the most essential property of the chosen particular solution (14) is its linearity in the variables p and q. This implies the linearity in those same variables of the expression $\{H_1, S_1\} + {}^1/_2 \{\{H_0, S_1\}, S_1\}$ which, by a time averaging analogous to (11), can be written as

$$\{H_1, S_1\} + {}^1/_2 \{\{H_0, S_1\}, S_1\} = K_2(J, p, q) + f^2(\psi, J)\, p + g^2(\psi, J)\, q$$

Substituting this representation into relation (10) for K_2, we obtain the equation that enables us to find S_2

$$p \frac{d\omega}{dJ} \frac{\partial S_2}{\partial q} - \omega \frac{\partial S_2}{\partial \psi} + f^2 p + g^2 q = 0$$

This differs from Eq.(13) only by inhomogeneous terms, and, consequently, its particular solution is given by a quadrature analogous to (14). By induction we get that in any approximation the components of the Lie generator are given by the formula

$$S_k = \frac{1}{\omega_n} \int \left[f^k \left(\omega \frac{\psi_n}{\omega_n} + r, J \right) p + g^k \left(\omega \frac{\psi_n}{\omega_n} + r, J \right) q \right] d\psi_n + \frac{1}{\omega_n{}^2}\, p \frac{d\omega}{dJ} \iint g^k \left(\omega \frac{\psi_n}{\omega_n} + r, J \right) d\psi_n{}^2, \quad r = \psi - \omega \frac{\psi_n}{\omega_n} \tag{15}$$

As a result the Hamiltonian is reduced to the form

$$K(J, p, q, \varepsilon) = \omega(J)\, p + \varepsilon [P_1(J)\, p + Q_1(J)\, q] + \varepsilon^2 [P_2(J)\, p + Q_2(J)\, q] + \ldots$$

in which there is no dependence on ψ, and which is linear in p and q.

The equations of motion of the system with the new Hamiltonian are

$$\psi^{\cdot} = \frac{\partial K}{\partial p} = \omega(J) + \varepsilon P_1(J) + \varepsilon^2 P_2(J) + \ldots, \qquad J^{\cdot} = \frac{\partial K}{\partial q} = \varepsilon Q_1(J) + \varepsilon^2 Q_2(J) + \ldots \tag{16}$$

The motion-separation problem has thus been solved. To find the solution of system (1) we should substitute the solution of system (16) into the replacement Eqs.(6) which, taking (5) and (9) into account, can be written to within terms of the third order of smallness inclusive

$$\varphi = \psi + \varepsilon \frac{\partial S_1}{\partial p} + \varepsilon^2 \left(\frac{\partial S_2}{\partial p} + \frac{1}{2} \left\{ \frac{\partial S_1}{\partial p}, S_1 \right\} \right) + \varepsilon^3 \left(\frac{\partial S_3}{\partial p} + \frac{1}{2} \left\{ \frac{\partial S_2}{\partial p}, S_1 \right\} + \frac{1}{2} \left\{ \frac{\partial S_1}{\partial p}, S_2 \right\} + \frac{1}{6} \left\{ \left\{ \frac{\partial S_1}{\partial p}, S_1 \right\}, S_1 \right\} \right)$$
$$I = J + \varepsilon \frac{\partial S_1}{\partial q} + \varepsilon^2 \left(\frac{\partial S_2}{\partial q} + \frac{1}{2} \left\{ \frac{\partial S_1}{\partial q}, S_1 \right\} \right) + \varepsilon^3 \left(\frac{\partial S_3}{\partial q} + \frac{1}{2} \left\{ \frac{\partial S_2}{\partial q}, S_1 \right\} + \frac{1}{2} \left\{ \frac{\partial S_1}{\partial q}, S_2 \right\} + \frac{1}{6} \left\{ \left\{ \frac{\partial S_1}{\partial q}, S_1 \right\}, S_1 \right\} \right) \tag{17}$$

Since replacement (17), just as the Eqs.(16), does not contain the variables p and q, the increase in the problem's dimensions at the price of introducing the adjoint variables remains finite and does not lead to any complication at any stage of the solution.

The motion-separation method constructed above relates to the so-called nonresonance case which is characterized by the fact that the time averaging introduced in (11) is identical with the space averaging

$$K_i{}^*(J, p, q) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} [F^i(\psi, J)\, p + G^i(\psi, J)\, q]\, d\psi_1 \ldots d\psi_n \tag{18}$$

In this case the expansion coefficients of the Lie generator (15) turn out to be bounded functions of ψ, while $K_i(J, p, q)$ are continuous functions of J. Here the computation of averages of type (11) can be replaced by computations by formula (18).

A resonance, to a first approximation, in a linear condition on the frequency

$$\Lambda_0 \omega(J) = 0 \tag{19}$$

for which the time average is not the same as the space average: $K_1(J, p, q) \neq K_1{}^*(J, p, q)$, i.e., conditions (19) define a surface of discontinuity in the space of slow variables J of the time average $K_1(J, p, q)$ as a function of J. In (19) Λ_0 is a $\nu \times n$-matrix with integral coefficients ($\lambda = \mathbf{rank}\, \Lambda_0 < n$, λ is the multiplicity of the resonance).

The corresponding condition for the function K_i is called resonance in the i-th approximation. A system is said to be resonant up to the i-th approximation, inclusive, if all

$K_1, K_2, \ldots, K_i$ are continuous. We emphasize that an arbitrary relation (19) is not called a resonance, but only the one which is connected with the discontinuity properties of the functions $K_i(J, p, q)$.

If the resonance case, or one close to it, holds, then the quality of the asymptotic solutions deteriorates due to the appearance in (15) of either secular terms or small denominators. In this case it is necessary, before taking the next step (if the resonance is in the i-th approximation, then this step is connected with the computation of $K_i(J, p, q)$), to regularize the problem by reducing it to the nonresonance case. The regularization of the problem is effected as follows. Suppose that we are dealing with the i-th-order resonance and that the motion-separation procedure being considered has already been implemented up to the $(i-1)$-th order, inclusive. The transformed Hamiltonian has the form

$$K = \omega(J)p + \varepsilon K_1(J, p, q) + \ldots + \varepsilon^{i-1}K_{i-1}(J, p, q) + \varepsilon^i[F^i(\psi, J)p + G^i(\psi, J)q] + \ldots \tag{20}$$

The time average

$$K_i = \lim_{\theta\to\infty}\frac{1}{\theta}\int_0^\theta [F^i(\omega\theta, J)p + G^i(\omega\theta, J)q]\,d\theta$$

is a discontinuous function of J on surface (19), which confirms resonance in the i-th approximation. (If ω is independent of J, then the presence of resonance is connected with the discontinuity of K_i with respect to ω).

If the surface indicated does not pass through the domain of those variables J of interest to us, then the case can be treated as a nonresonance one, and no regularization whatever is required to continue the procedure. If, however, we are interested in the system's behaviour in a small neighbourhood of the resonance surface

$$\Lambda_0\omega(J) = \varepsilon\Delta(J) \tag{21}$$

then we should carry out the canonical replacement $(\psi, J, p, q) \to (\gamma, L, r, s)$ using the formulas

$$\gamma = \Lambda\psi, \quad p = \Lambda^T r, \quad J = L, \quad q = s, \quad \Lambda = \begin{Vmatrix} E & 0 \\ & \Lambda_0 \end{Vmatrix} \tag{22}$$

where Λ is a block $n\times n$ -matrix and E is the unit $(n-\lambda)\times(n-\lambda)$ -matrix (without loss of generality we assume $\lambda = \nu$). Replacement (22) is canonical since it is produced by a generating function: $R(\psi, J, r, s) = r^T\Lambda_0\psi + s^TJ$. The scalar products in (20) of type ωp can be understood as matrix products by agreeing to take the first vector in the product as a row and the second as a column. As a result of replacement (22) the Hamiltonian (20) acquires the form

$$K = \omega(J)\Lambda^T r + \ldots + \varepsilon^{i-1}K_{i-1}(J, \Lambda^T r, q) + \varepsilon^i[F^i(\Lambda^{-1}\gamma, J)\Lambda^T r + G^i(\Lambda^{-1}\gamma, J)q] + \ldots \tag{23}$$

We rewrite the first term of this expression in the coordinate-wise form

$$\omega(J)\Lambda^T\mathbf{r} = \omega_1 r_1 + \ldots + \omega_{n-\lambda}r_{n-\lambda} + \varepsilon(\Delta_1 r_{n-\lambda+1} + \ldots + \Delta_\lambda r_n)$$

We see that the last λ pieces of the terms have an order of smallness ε and must be referred to the next term in expansion (23). This signifies that the number of fast variables in the system has decreased and become equal to $n-\lambda$. The resonance indicated has been eliminated, and we can apply the nonresonance procedure presented above to Hamiltonian (23). The averaging should be carried out over the remaining fast variables $\gamma_1, \ldots, \gamma_{n-\lambda}$.

Let us explain the meaning of the "near-resonance" condition (21). If ω is independent of J, then $\varepsilon\Delta$ is a constant, called a detuning, depending on the system's motion, and can be chosen arbitrarily small. If ω depends on J, then in times of the order ε^{-1} the variables J can be changed by a finite amount, which leads to the vector $\Lambda_0\omega(J)$ being changed by a finite amount. The formal introduction of a small parameter in (21) is carried out depending on the actual problem posed and, in particular, can be implemented thus. Suppose we are studying motions close to steady-state: $J = J_0 = \mathrm{const}$, and let $\Lambda_0\omega(J_0) = 0$ be the resonance condition. We carry out the canonical replacement: $J = J_0 + \varepsilon\alpha$; $q = \varepsilon^{-1}\beta$. This replacement does not change the structure of the Hamiltonian since $Q_1(J_0) = \ldots = Q_i(J_0) = 0$ and the order of all terms with respect to ε are preserved. Here the near-resonance condition takes the necessary formal form: $\Lambda_0\omega(J_0 + \varepsilon\alpha) = \varepsilon\Delta(\alpha, \varepsilon)$.

We note two cases where the proposed motion-separation method can be simplified.

First case. System (1) is already Hamiltonian. Its Hamiltonian $H = H_0(I) + \varepsilon H_1(\varphi, I)$. Obviously, now we do not need to introduce adjoint momenta and the above-mentioned Hamiltonian transformation procedure remains. Formulas (10) become

$$K_0 = H_0(J), \quad K_1 = -\omega \frac{\partial S_1}{\partial \psi} + H_1(\psi, J)$$

$$K_2 = -\omega \frac{\partial S_2}{\partial \psi} + \{H_1, S_1\} + \frac{1}{2}\{\{H_0, S_1\}, S_1\}, \ldots \quad \left(\omega = \frac{dH_0}{dJ}\right)$$

All the subsequent calculations connected with picking out the average and solving Eq.(13), which in this case is materially simplified, remain as before. The transformed Hamiltonian is independent of ψ, and the method proves to be equivalent to Poincaré's method /4/ with the sole difference that the Lie-generator technique is used instead of the generating-function technique.

Second case. System (1) is Hamiltonian, but its Hamiltonian is non-autonomous and depends periodically on time. This case can be reduced to the preceding one by introducing a new variable $\varphi_{n+1} = \omega_{n+1}t$ and the momentum adjoint to it (the term $\omega_{n+1}I_{n+1}$ should be added on to the Hamiltonian).

Example. The Duffing equation

$$x'' + x + 8\varepsilon x^3 = \varepsilon\mu \sin(3 + \varepsilon\Delta)\, t$$

after the replacement $x = I\sin\varphi_1,\ x' = I\cos\varphi_1$, reduces to a system whose Hamiltonian is

$$H(\varphi, I, u, v, \varepsilon) = u_1 + 3u_2 + \varepsilon(8I^2\sin^4\varphi_1 - \mu I^{-1}\sin\varphi_1\sin\varphi_2)u_1 +$$
$$\varepsilon\Delta u_2 + \varepsilon(-8I^3\sin^3\varphi_1\cos\varphi_1 + \mu\cos\varphi_1\sin\varphi_2)\, v$$

Formally replacing (φ, I, u, v) by (ψ, J, p, q) in this expression, in accordance with formula (11), we find $K_1(J, p_1, p_2, q) = 3p_1J^2 + \Delta p_2$. Equation (13) takes the form

$$-\frac{\partial S_1}{\partial \psi_1} - 3\frac{\partial S_1}{\partial \psi_2} + [J^2(\cos 4\psi_1 - 4\cos 2\psi_1) - \mu J^{-1}\sin\psi_1\sin\psi_2]\, p_1 +$$
$$(-8J^3\sin^3\psi_1\cos\psi_1 + \mu\cos\psi_1\sin\psi_2)\, q = 0$$

According to formula (14) its solution is

$$S_1 = \left\{J^2\left(\frac{1}{4}\sin 4\psi_1 - 2\sin 2\psi_1\right) - \frac{1}{8}\mu J^{-1}[2\sin(\psi_2 - \psi_1) - \sin(\psi_2 + \psi_1)]\right\}p_1 -$$
$$\left\{\frac{J^3}{4}(\cos 4\psi_1 - 4\cos 2\psi_1) + \frac{\mu}{8}[2\cos(\psi_2 - \psi_1) + \cos(\psi_2 + \psi_1)]\right\}q$$

Substituting H_1 and S_1 into (10) we see that $K_2(J, p, q) \neq K_2^*(J, p, q)$, i.e., the time average is not the same as the space average (both averages are computed from the expression $\{H_1, S_1\} + {}^1/_2\{\{H_0, S_1\}, S_1\}$). This signifies that we are dealing with resonance in the second approximation and that to continue the procedure we should regularize the problem. For regularization we make the canonical replacement (22)

$$\gamma_1 = \psi_1,\ \gamma_2 = 3\psi_1 - \psi_2,\ p_1 = r_1 + 3r_2,\ p_2 = -r_2$$

After which the averaging is carried out with respect to the variable γ_1

$$K_2(\gamma_2, J, r_1, r_2, q) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}\{K_1, S_1\} + \frac{1}{2}\{\{H_0, S_0\}, S_0\}\right)d\gamma_1 =$$
$$(r_1 + 3r_2)\left(-\frac{51}{4}J^4 + \frac{3}{8}\mu J\cos\gamma_2\right) + \frac{3}{8}q\mu J^2\sin\gamma_2$$

Restricting ourselves to the first two approximations, we obtain the Hamiltonian

$$K = r_1 + \varepsilon[3(r_1 + 3r_2)J^2 - r_2\Delta] +$$
$$\varepsilon^2\left[(r_1 + 3r_2)\left(-\frac{51}{4}J^4 + \frac{3}{8}\mu J\cos\gamma_2\right) + \frac{3}{8}q\mu J^2\sin\gamma_2\right]$$

in correspondence with which we have a system in which the slow variables have been separated from the fast up to second order, inclusive, with due regard to the existing resonance

$$\gamma_1' = \frac{\partial K}{\partial r_1} = 1 + 3\varepsilon J^2 + \varepsilon^2\left(-\frac{51}{4}J^4 + \frac{3}{8}\mu J\cos\gamma_2\right)$$
$$\gamma_2' = \frac{\partial K}{\partial r_2} = \varepsilon(9J^2 - \Delta) + 3\varepsilon^2\left(-\frac{51}{4}J^4 + \frac{3}{8}\mu J\cos\gamma_2\right)$$
$$J' = \frac{\partial K}{\partial q} = \frac{3}{8}\varepsilon^2\mu J^2\sin\gamma_2$$

By formulas (17) we find the connection between the old variables and the new

$$\varphi_1 = \psi_1 + \varepsilon\frac{\partial S_1}{\partial p_1} = \gamma_1 + \left[\frac{J^2}{4}\sin 4\gamma_1 - 2J^2\sin 2\gamma_1 - \frac{\mu}{4}J^{-1}\sin(2\gamma_1 - \gamma_2) + \frac{\mu}{8}J^{-1}\sin(4\gamma_1 - \gamma_2)\right]\varepsilon$$

$$I = J + \varepsilon\frac{\partial S_1}{\partial q} = J + \left\{-\frac{J^3}{4}\cos 4\gamma_1 + J^3\cos 2\gamma_1 - \frac{\mu}{8}[2\cos(2\gamma_1 - \gamma_2) + \cos(4\gamma_1 - \gamma_2)]\right\}\varepsilon$$

Or, in the original variable

$$x = I\sin\varphi_1 = \left(J - \frac{3}{2}\varepsilon J^3\right)\sin\gamma_1 - \frac{\varepsilon}{8}[2J^3\sin 3\gamma_1 + \mu\sin(3\gamma_1 - \gamma_2)]$$

In the problem considered we could manage without the introduction of adjoint variables, since the original system can be written immediately in Hamiltonian form.

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ON THE CORNER POINTS OF THE BOUNDARIES OF REGIONS OF ATTAINABILITY*

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Some properties of the boundaries of the regions of attainability of linear unsteady systems with a single control (perturbation) function that has values on the segment are studied. It is established that for fairly small time intervals the boundary of the attainment region has conical corner points, edges and faces. The conditions for which the distance of the conical corner points from the origin of coordinates is a maximum are established.

1. The regions of attainability of controllable (perturbable) systems were studied in /1—8/ and elsewhere. The interest in investigating such regions is connected, for instance, with the Bulgakov problem of the accumulation of perturbations /9/. The method of attainment regions was used when constructing optimal-control theory /2—4/ and the theory of games /5, 6/. Their determination is important in many applications. In /10, 11/ it was proposed to evaluate these regions by using ellipsoids. In the present paper certain statements are proved on the presence or absence of corner points ("tapered" points /5/) on the boundary of the attainment regions, on the properties of such points, and on their extremal properties. The presence of corner points is indicative of the limited nature of the approach in which the boundaries are approximated by smooth surfaces. The question of the extremal properties of boundary points arises when determining the control that removes the system farthest away from the origin of coordinates.

Consider a system defined by the linear matrix differential

$$dx/dt = A(t)x + b(t)u, \quad |u(t)| \leqslant 1 \tag{1.1}$$

where $x, A(t)$ and $b(t)$ are matrices of the type $(n\times 1), (n\times n)$ and $(n\times 1)$, respectively, and $u(t)$ is the control (perturbing) function bounded in absolute value and piecewise continuous; the set of such functions will be denoted by Ω. We will assume that the matrix elements $A(t)$ and $b(t)$ have continuous derivatives up to the $(n-1)$-th order for all t.

The solution of system (1.1) at the instant $t = T$ when $x(t_0) = 0$ is described by the integral

$$x(T) = \int_{t_0}^{T} \theta(T)\theta^{-1}(\tau)b(\tau)u(\tau)\,d\tau \tag{1.2}$$

*Prikl.Matem.Mekhan., Vol.47, No.4, pp.556-574, 1983